



Let (\mathcal{R}, d) be the metric space under consideration. $d(x, y) = |x - y|$ is the distance measure ($x, y \in \mathcal{R}$).

Let E be a subset of \mathcal{R}

Let $I_o = \{x \mid a < x < b; \text{ where } a, b, x \in X \}$.

This is also denoted by (a, b) . Examples: $(0, 1)$, $(2, 6)$

Some observations about I_o :

- does not have its boundary points
- has no minimum and has no maximum
- has infinite number of elements (points)

Let $I_c = \{x \mid a \leq x \leq b; \text{ where } a, b, x \in X \}$

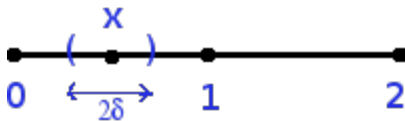
This is also denoted by $[a, b]$. Examples: $[1, 3]$, $[1, 5]$

Some observations about I_c :

- has both of its boundary points (a and b).
- has a minimum (a) and a maximum (b)
- has infinite number of elements (points)

A **neighborhood** of a point $x \in E$ denoted by $N_\delta(x)$ is the set of points $(x - \delta, x + \delta)$ where $\delta > 0$.

$$N_\delta(x) = \{y \mid |x - y| < \delta; \delta > 0\}$$



point types

A point $x \in E$ could have:

1. a set of points in every neighborhood
 - these points (other than x) from each such neighborhood set could also be a member of E , then x is known as **limit point** of E
 - none of the points (other than x) from most such neighborhood sets are a member of E , then x is known as **isolated** of E . There might be some δ values for which the neighborhood sets have points in common with E , but not for all δ values
2. some neighborhood
 - where every one of these neighboring points could belong to E , then x is known as **interior point** of E

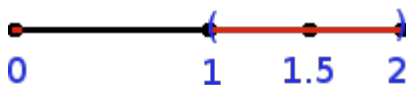
A point $x \in X; \notin E$ might have:

1. a set of points in every neighborhood
 - these points (other than x) from each such neighborhood set could also be a member of E , then x is known as **limit point** of E

examples of point types

Here are some examples with \mathcal{R} as the metric space under consideration:

- Let $E_1 = (1,2) \cup \{0\}; E_1 \subset \mathcal{R}$



- 0 is an isolated point of E_1

0 is far away from rest of the members of E_1 . The possible δ values for which N_δ has some commonality with \mathcal{R} is $1 < \delta$, but it is not the case for every δ .
- every $x \in (1,2)$ is an interior point of E_1

For every point in the range (1,2), there is some $\delta > 0$ such that N_δ is a subset of E_1 . Points in the middle of the range (1,2) have larger such neighborhoods compared to points closer to end points (1 or 2).
- every $x \in (1,2)$ is a limit point of E_1

For every point in the range (1,2), the neighborhood (for every $\delta > 0$) has some points in common with E_1
- 1 and 2 are limit points too, but they are not contained in E_1

- Let $E_2 = [1,2] \cup \{0\}; E_2 \subset \mathcal{R}$



- 0 is an isolated point of E_2

0 is far away from rest of the members of E_2 . The possible δ values for which N_δ has some commonality with \mathcal{R} is $1 < \delta$, but it is not the case for every δ .
- every $x \in (1,2)$ is an interior point of E_2 . 1 and 2 are not interior points

For every point in the range (1,2), there is some $\delta > 0$ such that N_δ is a subset of E_2 . Points in the middle of the range (1,2) have larger such neighborhoods compared to

points closer to end points (1 or 2). The end points 1 and 2 does not have a single δ such that the neighborhood is a subset of E_2 .

- every $x \in [1,2]$ is a limit point of E_2

For every point in the range $[1,2]$, the neighborhood (for every $\delta > 0$) has some points in common with E_2

- Taking the whole space \mathcal{R}
 - every $x \in \mathcal{R}$ is an interior point of \mathcal{R}
 - every $x \in \mathcal{R}$ is a limit point of \mathcal{R} and is contained in \mathcal{R}
- Let $E_3 = \{1, 2, 1.5, 9\}$; $E_3 \subset \mathcal{R}$
 - each point of E_3 is an isolated point
 - E_3 has no limit points
 - E_3 has no interior points
 - 1 and 2 are limit points, and they are members of E_1

Open sets/Closed sets

- A subset E of a metric space is termed **open** if every point of E is an interior point of E .
 $\forall x \in E, \exists \delta > 0$ such that $(x-\delta, x+\delta) \subseteq E$
examples:
 - (a,b) in \mathcal{R} where $a,b \in \mathcal{R}$
 - $\{0\} \cup (1,2)$
- A subset E of a metric space (X) is termed **closed** if every limit point of E is contained in E .
 E is said to be open if $X \setminus E$ is open.
examples:
 - $[a,b]$ in \mathcal{R} where $a,b \in \mathcal{R}$
 - a finite subset of \mathcal{R} - $\{1,4,5\}$
- A subset E of a metric space is termed **clopen** if the subset is closed and open
examples:
 - the whole \mathcal{R}
 - the null set \emptyset
- A subset (such as $(1,2]$) could be neither open nor closed

why bother?

- Some questions of interest:
 - How deep the points are situated in a subset of a space?

- How close are the points in a subset to a given point?
- A property of a metric space or a map between metric spaces that can be described solely using open subsets (or closed subsets) is known as *topological property*. Continuity is such a topological property- functions that ensure that pre-image of an open subset (in co-domain) at a point is open (in domain) is continuous at that point.

To explore

- How to show or infer (computationally) whether a given set is open (or closed)? Will such an exercise be of any value? Or is the formalism useful (only) for making other concepts compactly expressible?