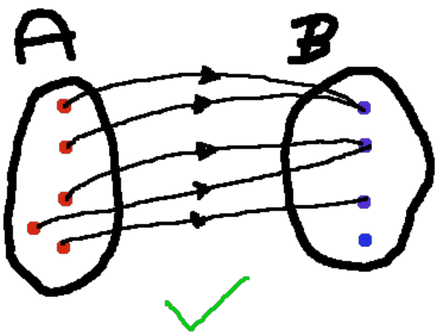
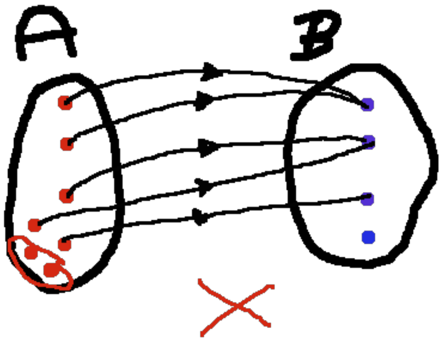
A ***function*** (also known as a ***map***) $\alpha : A \rightarrow B$ with ***domain*** A (set) and ***codomain*** (also known as ***range)*** B (set) is defined when,

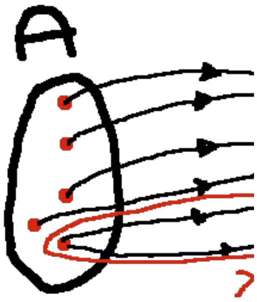
for each $a \in A$, there is a unique $b \in B$ such that $a\alpha = \alpha(a) = b$.

$\forall\ a \in A, \exists !\ b \in B$ such that $a\alpha = \alpha(a) = b$





(invalid: unmapped entries)



(invalid: nonunique mapping)

The above definition needs the mapping to be defined for every element in the domain. The above definition also restricts the mapping to be single-valued. With these restrictions it is ensured that the map is defined for every element of the set A and every element of A is mapped to a single element in B under the given map.

Image & Pre-image

Image

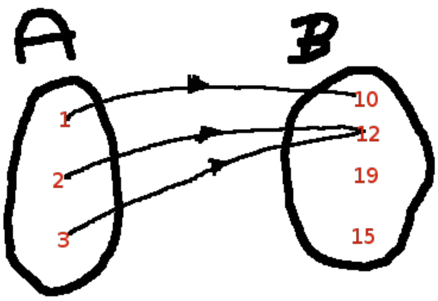
of an element a ($\in$ A)under the map α is the corresponding b ($\in$ B).

- $\forall\ a \in A$, $Img(a) = \{b \mid \alpha(a) = b, b \in B\}$
- (I_1) Every element a of A will have only one element as image
- (I_2) Let $Img_\cap = \{\cap Img(a) \mid a \in A\}$. $Img_\cap$ may not be empty
- (I_3) Let $Img_\cup = \{\cup Img(a) \mid a \in A\}$. $Img_\cup$ may not equal B

Pre-image (or ***inverse-image***)

of an element b is the set of elements of A for which b is the corresponding element under the map α.

- $\forall\ b \in B$, $Pre\text{-}Img(b) = \{a \mid \alpha(a) = b, a \in A\}$
- For a given b ($\in$ B)
  - ($P_{1.1}$) Pre-Img(b) may be empty
  - ($P_{1.2}$) Pre-Img(b) may have only one element
  - ($P_{1.3}$) Pre-Img(b) may have more than one elements
- (P_2) $\{\cap Pre\text{-}Img(b) \mid b \in B, Pre\text{-}Img(b) \neq \varnothing\}$ will be empty
- (P_3) $\{\cup Pre\text{-}Img(b) \mid b \in B\}$ will equal A



In the map shown here, A= {1, 2, 3} and B = {10, 12, 19, 15}.

Image of elements of A

- Image of 1 is {10}
- Image of 2 is {12}
- Image of 3 is {12}

Pre-image of elements of B

- Pre-image of 10 is {1}
- Pre-image of 12 is {2, 3}
- Pre-image of 19 is {}
- Pre-image of 15 is {}

Types of functions

With the above definition of function in place, are there maps with interesting properties? This question can be answered by exploring what parameters can be tweaked (without violating the definition) to specialize.

one-to-one or ***injection***

$\alpha : A \rightarrow B$ is injective when $\forall\ a_1, a_2 \in A, \alpha(a_1) = \alpha(a_2) \Rightarrow a_1 = a_2$

In general, many elements of the domain can be mapped to the same element in co-domain. This means that the intersection of Img() sets (of elements from the domain) may not be empty (see (I_2)). A map can be defined where the Img() set intersection is empty . Such a map where no two elements of the set A is mapped to same element in B is known as ***one-to-one*** or ***injection***. With this kind of map, *the "number of elements" of domain(A) is less than or equal to the "number of elements" co-domain(B)*.

This kind of map may be understood (or spotted) by specializing inthe follwoing ways:

- Specializing (I_2) - case where $Img_\cap$ is empty
- Specializing ($P_{1.2}$) - case where $P_{1.2}$ is true for all b ($\in$ B)

onto or ***surjection***

$\alpha : A \rightarrow B$ is surjective when $\forall\ b \in B, \exists\ a \in A$ such that $\alpha(a) = b$

In general, there might be some elements of B that pre-image set as empty (ie. no element of A is mapped to this element b). A special case of this is a map where every element of B has a non-empty pre-image. This kind of map is known as ***onto*** or ***surjection***. In this case, the *"number of elements" of the domain (A) will be equal or more than the "number of elements" of co-domain(B)*.

This kind of map may be understood (or spotted) by specializing in the following ways:

- Specializing (I_3) - case where $Img_{\cup}$ is equal to B
- Specializing ($P_{1.1}$) - case where $P_{1.1}$ is true for all b (∈ B)

bijective

$\alpha : A \rightarrow B$ is bijective when $\forall\, b \in B,\ \exists!\, a \in A$

It is possible that a map (from set A to set B) is *injective and surjective*. This kind of map is known as ***bijective***